COINCIDENCE THEORY IN ARBITRARY CODIMENSIONS: THE MINIMIZING PROBLEM

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Let $f_1, f_2 : M \longrightarrow N$ be two (continuous) maps between smooth connected manifolds M and N without boundary, of strictly positive dimensions m and n, resp., M being compact. We are interested in making the coincidence locus

$$C(f_1, f_2) := \{x \in M \mid f_1(x) = f_2(x)\}$$

as small (or simple in some sense) as possible after possibly deforming f_1 and f_2 by a homotopy.

Question. How large is the **m**inimum number of **c**oincidence **c**omponents

$$MCC(f_1, f_2) := \min\{\#\pi_0(C(f_1', f_2')) \mid f_1' \sim f_1, f_2' \sim f_2\} \ ?$$

In particular, when does this number vanish, i.e. when can f_1 and f_2 be deformed away from one another?

This is a very natural generalization of one of the central problems of classical fixed point theory (where $M = N$ and f_2 = identity map): determine the minimum number of fixed points among all maps in a given homotopy class (see [Br] and [BGZ], proposition 1.5). Note, however, that in higher codimensions $m - n > 0$ the coincidence locus is generically a closed $(m-n)$-manifold so that it makes more sense to count *pathcomponents* rather than points. Also the methods of (first order, singular) (co)homology will no longer be strong enough to capture the subtle geometry of coincidence manifolds.

In this lecture I will use the language of normal bordism theory (and a nonstabilized version thereof) to define and study lower bounds $N(f_1, f_2)$ (and $N^{\#}(f_1, f_2)$) for $MCC(f_1, f_2)$.

After performing an approximation we may assume that the map $(f_1, f_2) : M \to N \times N$ is smooth and transverse to the diagonal $\Delta = \{(y, y) \in N \times N \mid y \in N\}$. Then the coincidence locus

$$C \ = \ C(f_1, f_2) \ = \ (f_1, f_2)^{-1}(\Delta)$$

is a closed smooth $(m - n)$-dimensional manifold, equipped with

i) maps

$$\begin{tikzcd} & E(f_1,f_2) \arrow[d, "\mathrm{pr}"] \\ C \arrow[ur, "\tilde g"] \arrow[r, "g = \mathrm{incl}"'] & M \end{tikzcd} \qquad E(f_1, f_2) := \left\{ (x, \theta) \in M \times N^I \ \middle| \ \begin{matrix} \theta(0) = f_1(x); \\ \theta(1) = f_2(x) \end{matrix} \right\}$$

where $\widetilde{g}$ is the natural lifting which adds the constant path at $f_1(x) = f_2(x)$ to $g(x) = x \in C$; and

ii) a stable vector bundle isomorphism

$$\overline{g} \ : \ TC \oplus g^*(f_1^*(TN)) \ \cong \ g^*(TM)$$

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deduced from the isomorphism

$$\overline{g}^{\#} \;:\; \nu(C,M) \;\cong\; (f_1,f_2)^*(\nu(\Delta, N\times N)) \;\cong f_1^*(TN)\;|C$$

of (nonstable) normal bundles.

The triple $(C,\widetilde{g},\overline{g})$ gives rise to a welldefined bordism class

$$\widetilde{\omega}(f_1,f_2) \;:=\; [C,\widetilde{g},\overline{g}] \;\in\; \Omega_{m-n}(E(f_1,f_2);\;\widetilde{\varphi})$$

in the normal bordism group of such triples (here the virtual coefficient bundle is defined by

$$\widetilde{\varphi} \;:=\; pr^*(f_1^*(TN) - TM)\;;$$

e.g. if M and N are stably parallelized, then $\widetilde{\varphi}$ is trivial and we are dealing with (stably) framed bordism).

Keeping track also of the fact that C is a smooth submanifold of M with (non-stabilized) normal bundle described by $\overline{g}^{\#}$, we obtain a sharper invariant

$$\omega^{\#}(f_1,f_2) \;\in\; \Omega^{\#}(f_1,f_2)$$

which, however, lies in general only in a suitable bordism *set* (not group).

A crucial ingredient of both the $\widetilde{\omega}$- and the $\omega^{\#}$-invariant is the map $\widetilde{g}$. Indeed, the path space $E(f_1,f_2)$ has a very rich topology. Already its set $\pi_0(E(f_1,f_2))$ of pathcomponents can be huge – it corresponds bijectively to the so called Reidemeister set $R(f_1,f_2)$, a well-studied set-theoretic quotient of the fundamental group $\pi_1(N)$. This leads to a natural decomposition

$$C(f_1,f_2) \;=\; \coprod_{A\in\pi_0(E(f_1,f_2))} \widetilde{g}^{-1}(A)\quad .$$

Let $N(f_1,f_2)$, and $N^{\#}(f_1,f_2)$, resp., denote the corresponding number of nontrivial contributions by the various pathcomponents A of $E(f_1,f_2)$ to $\widetilde{\omega}(f_1,f_2)$ and $\omega^{\#}(f_1,f_2)$, resp.

Theorem 1. (i) *The integers* $N(f_1,f_2)$ *and* $N^{\#}(f_1,f_2)$ *depend only on the homotopy classes of* f_1 *and* f_2*;*
(ii) $N(f_1,f_2) = N(f_2,f_1)$ *and* $N^{\#}(f_1,f_2) = N^{\#}(f_2,f_1)$*;*
(iii) $0 \le N(f_1,f_2) \le N^{\#}(f_1,f_2) \le MCC(f_1,f_2) < \infty$*;*
(iv) *if* $m = n$ *then* $N(f_1,f_2) = N^{\#}(f_1,f_2)$ *coincides with the classical Nielsen number* (*which has a standard definition at least if both* M *and* N *are orientable or if* f_2 *is the identity map*).

Recall the decisive progress made by *J. Nielsen* on the classical minimizing problem when he decomposed fixed point sets into equivalence classes. In our interpretation this is just the decomposition of a 0-dimensional bordism class according to the pathcomponents of its target space. In higher (co)dimensions $m-n$ the map $\widetilde{g}$ into $E(f_1,f_2)$ and the twisted framing $\overline{g}^{\#}$ contain much more information. E.g. if $M = S^m$ and $n \ge 2$ then $\Omega^{\#}(f_1,f_2)$ can be identified with the homotopy group $\pi_m(S^n \wedge \Omega(N)^+)$, and $\omega^{\#}(f_1,f_2)$ is closely related to a Hopf-Ganea invariant. This allows us to reduce many aspects of our problem to questions in standard homotopy theory.

Details of definitions, proofs, and applications will be given elsewhere (compare e.g. [K 3] and [K 2]). Here we present just one sample result.

Theorem 2. *Let* N *be an odd-dimensional spherical space form* (*i.e. the quotient of* S^n *by a free action of a finite group*)*. Then we have for all* $f_1, f_2 : S^m \to N$*:*

$$MCC(f_1,f_2) = N^{\#}(f_1,f_2) = \begin{cases} 0 & \text{if } f_1 \sim f_2 \text{ or } m < n; \\ \#\pi_1(N) & \text{if } f_1 \not\sim f_2 \text{ and } m > 1; \\ |d^0(f_1) - d^0(f_2)| & \text{if } m = n = 1. \end{cases}$$

(*Here* $d^0(f_i) \in \mathbb{Z}$ *denotes the usual degree*).

Finally note that our approach applies also to over- and undercrossings of link maps into a manifold of the form $N \times \mathbb{R}$. This yields unlinking obstructions which often settle unlinking questions and which, in addition, turn out to distinguish a great number of different link homotopy classes. In certain cases they even allow a complete link homotopy classification. Moreover, our approach leads also to the notion of Nielsen numbers of link maps (cf. [K 4]).

References

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